Midterm 3.1-4.4

change of bases for subspaces Not covered on exam! Just $\mathbb{R}^n$

11/8-Lecture Notes

- To check for basis in $\mathbb{R}^3$ (if vectors are a) → check if they are linearly independent
  - ↳ Put in an augmented matrix & check for free variables

Ex] Is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ a basis for $\mathbb{R}^3$?

Yes, because $\mathcal{B}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ consists of vectors that are linearly independent and therefore span all of $\mathbb{R}^3$

Notation: $\begin{bmatrix}a\\b\\c\end{bmatrix}_{\mathcal{B}}=a\begin{bmatrix}1\\1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}+c\begin{bmatrix}-1\\1\\1\end{bmatrix}$, and more generally if $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_n\}$ is a basis for $\mathbb{R}^n$ then $\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}_{\mathcal{B}}=x_1\vec{b}_1+\dots+x_n\vec{b}_n$

Ex] $\begin{bmatrix}-2\\1\\3\end{bmatrix}_{\mathcal{B}}=(-2)\begin{bmatrix}1\\1\\0\end{bmatrix}+\begin{bmatrix}0\\1\\1\end{bmatrix}+3\begin{bmatrix}-1\\1\\1\end{bmatrix}=\begin{bmatrix}-5\\2\\4\end{bmatrix}$

Notice: $\begin{bmatrix}1&0&-1\\1&1&1\\0&1&1\end{bmatrix}\begin{bmatrix}-2\\1\\3\end{bmatrix}=\begin{bmatrix}-5\\2\\4\end{bmatrix}$

Theorem: Let $\vec{x}$ be a vector, $\mathcal{B}=\{\vec{u}_1,\dots,\vec{u}_n\}$ be a basis for $\mathbb{R}^n$, $U=[\vec{u}_1,\dots,\vec{u}_n]$ then:

(a) $\vec{x}=U\vec{x}_{\mathcal{B}}$

(b) $\vec{x}_{\mathcal{B}}=U^{-1}\vec{x}$

Ex] $U=\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\1&1&1&0&1&0\\0&1&1&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&2&-1&1&0\\0&1&1&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&0&1&-1\\0&1&0&1&-1&2\\0&0&1&-1&1&-1\end{array}\right]\Rightarrow U^{-1}=\begin{bmatrix}0&1&-1\\1&-1&2\\-1&1&-1\end{bmatrix}$

$\begin{bmatrix}0&1&-1\\1&-1&2\\-1&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}y-z\\x-y+2z\\-x+y-z\end{bmatrix}_{\mathcal{B}}$

Ex] $\mathcal{B}_1=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$. $\mathcal{B}_2=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Write $\begin{bmatrix}1\\-1\\2\end{bmatrix}_{\mathcal{B}_2}$ in terms of $\mathcal{B}_1$.

$\begin{bmatrix}1\\-1\\2\end{bmatrix}_{\mathcal{B}_2}=(1)\begin{bmatrix}1\\2\\3\end{bmatrix}+(-1)\begin{bmatrix}1\\1\\1\end{bmatrix}+(2)\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1&1&0\\2&1&0\\3&1&1\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}0\\1\\4\end{bmatrix}$ ⇒ vector w.r.t. standard basis

Theorem: Let $\vec{x}$ be a vector, $\mathcal{B}_1=\{\vec{u}_1,\dots,\vec{u}_n\}$ & $\mathcal{B}_2=\{\vec{v}_1,\dots,\vec{v}_n\}$ be bases of $\mathbb{R}^n$. $U=[\vec{u}_1,\dots,\vec{u}_n]$ & $V=[\vec{v}_1,\dots,\vec{v}_n]$

$$\left.\begin{array}{l}\text{(a) } \vec{x}_{\mathcal{B}_1}=U^{-1}V\vec{x}_{\mathcal{B}_2}\\ \text{(b) } \vec{x}_{\mathcal{B}_2}=V^{-1}Ux_{\mathcal{B}_1}\end{array}\right\}\text{ we call } U^{-1}V \text{ etc. a "change of basis" matrix}$$

Ex] Write each vector in $\mathcal{B}_1$ in terms of $\mathcal{B}_2$

$\left[\begin{array}{ccc|ccc}1&1&0&1&0&-1\\2&1&0&1&1&1\\3&1&1&0&1&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&1&0&1&0&-1\\0&-1&0&-1&1&3\\0&-2&1&-3&1&4\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&0&1&2\\0&1&0&1&-1&-3\\0&0&1&-1&-1&10\end{array}\right]$

$$\left.\begin{array}{l}\begin{bmatrix}1\\1\\0\end{bmatrix}=(0)\begin{bmatrix}1\\2\\3\end{bmatrix}+(1)\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}\\ \begin{bmatrix}0\\1\\1\end{bmatrix}=1\begin{bmatrix}1\\2\\3\end{bmatrix}-1\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}\\ \begin{bmatrix}-1\\1\\1\end{bmatrix}=2\begin{bmatrix}1\\2\\3\end{bmatrix}-3\begin{bmatrix}1\\1\\1\end{bmatrix}-2\begin{bmatrix}0\\0\\1\end{bmatrix}\end{array}\right\}\quad \begin{array}{l}\vec{x}=\begin{bmatrix}a\\b\\c\end{bmatrix}_{\mathcal{B}_1}\\ \vec{x}_{\mathcal{B}_2}=\begin{bmatrix}0&1&2\\1&-1&5\\-1&-1&-2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}\end{array}$$

$[U|V]\sim[I|U^{-1}V]$